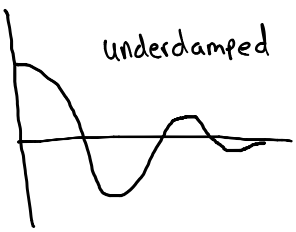
Vibrations

sinusoidally – forced damped harmonic oscillator

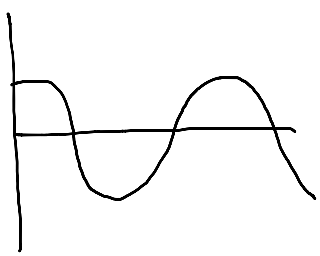
$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega_f t)$$

transient solution

or



and steady state solution

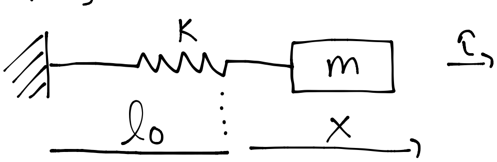


fix amplitude and parameters, biggest peak depends on damping factor, not just $\sqrt{\frac{k}{m}}$

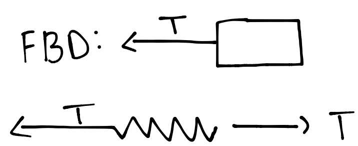
resonance and all that

Core concept: $m\ddot{x} + kx = 0$ $\longrightarrow$ harmonic oscillator

- spring-mass system



FBD:



LMB: $\vec{F} = m\vec{a}$

Dot both sides by $\hat{\imath}$ for 1-D

$$-T\hat{\imath} = m\ddot{x}\hat{\imath}$$

$$-T\hat{\imath}\cdot\hat{\imath} = m\ddot{x}\,\hat{\imath}\cdot\hat{\imath} \longrightarrow -T = m\ddot{x}$$

$$m\ddot{x} = -kx$$

$$\boxed{\ddot{x} = -\frac{k}{m}x \longrightarrow m\ddot{x} + \frac{k}{m}x = 0}$$

$\dot{x} = v$

$\dot{v} = \frac{-k}{m} x$ } ready for Matlab

Solution to diff eq: $x = A\cos(\omega t) + B\sin(\omega t)$

ω = natural frequency, damped frequency, etc.

$\omega_0 = \sqrt{\frac{k}{m}}$ for our equation

How do you show it's the general solution? You can satisfy any initial condition with the solution

Less sophisticated guess: $x = e^{\lambda t}$

$$m\lambda^2 + c\lambda + k = 0$$

$$\lambda = \dots \pm \sqrt{\frac{k}{m}}$$

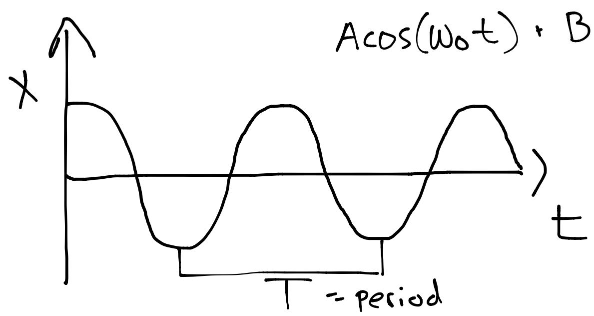
$$x = e^{i\omega_0 t}$$

$x = A\,\mathbb{R}(e^{i\omega t}) + B\,\mathrm{Im}(e^{i\omega t})$

or

$x = Ce^{i\omega t} + De^{-i\omega t}$

$A\cos(\omega_0 t) + B\sin(\omega_0 t)$



$\omega_0 T = 2\pi$

$T = \frac{2\pi}{\omega_0}$

$\omega_0 = \frac{2\pi}{T}$

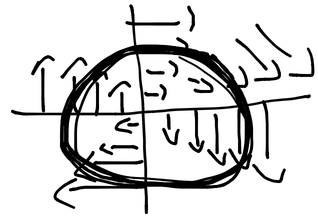
ω_0 = angular frequency

- angle per unit time

* other frequency $f = \frac{1}{T}$ → cycles per unit time

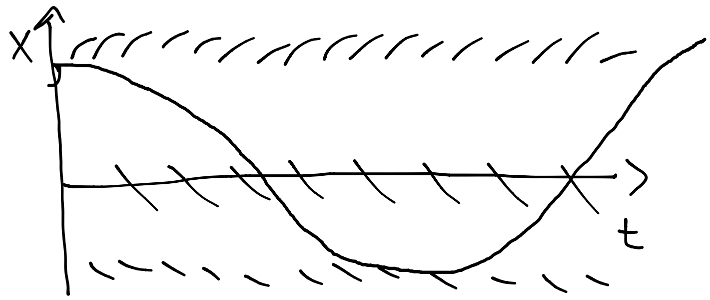
Phase Plane

$\dot{x} = v$
$\dot{v} = -\frac{k}{m} x$



ellipse

$\ddot{x} = -\frac{k}{m} x$



If you notice energy is conserved, amplitude will be constant

Andy walks back and forth

Energy

$m\ddot{x} + kx = 0$ multiply by $\dot{x}$

$m\ddot{x}\dot{x} + kx\dot{x} = 0$ $\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$

$\boxed{E_{tot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}}$ → constant amplitude oscillations

Start with conservation of energy

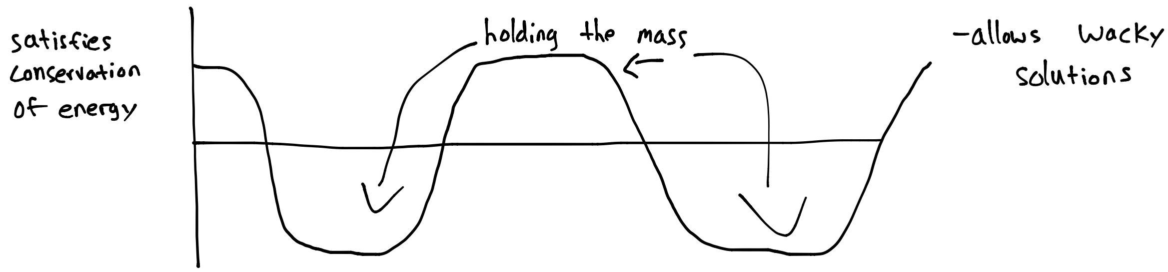
- Derive O.D.E.

$E_{tot} = E_K + E_P = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$\frac{d}{dt}(E_{tot}) = 0$ $m\ddot{x}\dot{x} + kx\dot{x} = 0$

if $\dot{x} \neq 0$ → $m\ddot{x} + kx = 0$

if $\dot{x} = 0$ → we don't know



Lagrange Equations

1 degree of freedom (dof)

$$\text{L.E.} = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\mathcal{L}}{\partial \dot{q}} = 0 \qquad \mathcal{L} = \mathcal{L}(q, \dot{q}) = E_K - E_P$$

$$E_P = \frac{1}{2} K x^2 \qquad x = q$$

$$E_K = \frac{1}{2} m \dot{x}^2$$

$$\mathcal{L} = \underbrace{E_K - E_P}_{\text{"}T\text{"} - \text{"}V\text{"}} = \underbrace{\frac{1}{2} m \dot{x}^2 - \frac{1}{2} K x^2}_{\text{"}\mathcal{L}(x, \dot{x})\text{"}}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$-Kx - \frac{d}{dt}(m\dot{x}) = 0 \qquad Kx + m\ddot{x} = 0$$